The algorithm behind `gensolve`

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1. Problem statement

Consider the underdetermined linear system

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{B}, \quad \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{X} \in \mathbb{R}^{n\times k}, \boldsymbol{B} \in \mathbb{R}^{m\times k} \tag{1}$$

with $m < n$. We seek a general solution of the from

$$\boldsymbol{X} = \boldsymbol{X}_\mathrm{p} + \boldsymbol{H}\boldsymbol{z}, \quad \boldsymbol{z} \text{ arbitrary,} \tag{2}$$

where $\boldsymbol{X}_\mathrm{p}$ is a specific solution of eq. (1) and the columns of $\boldsymbol{H}$ span the nullspace of $\boldsymbol{A}$. We assume $\boldsymbol{A}$ to have full rank, i.e. linearly independent rows, and hence a solution to exist and $\boldsymbol{H} \in \mathbb{R}^{n\times(n-m)}$.

2. Algorithm

The alogrithm implemented in `gensolve` is based on [1], which focuses on finding the nullspace of sparse matrices. However, the ABS algorithms it is derived from are actually concerned with solving equation systems, so it is relatively easy to extend the algorithm to also compute a specific solution.

In the follwowing, let $I_1, \dots, I_m$ be a permutation of $1, \dots, m$ and let $\boldsymbol{a}_{I_i}^T$ and $\boldsymbol{b}_{I_i}^T$ denote the I_i-th row of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. The permutation $I_1, \dots, I_m$ is chosen such that the number of non-zero elements in the rows $\boldsymbol{a}_{I_1}^T, \dots, \boldsymbol{a}_{I_m}^T$ is increasing.

Start with the initiaization

$$\boldsymbol{H}^{(0)} = \boldsymbol{I} \in \mathbb{R}^{n\times n} \tag{3}$$

$$\boldsymbol{X}^{(0)} = \boldsymbol{0} \in \mathbb{R}^{n}. \tag{4}$$

Then for $i = 0, \dots, m-1$ compute

$$s^{(i)T} = a^T_{I_{i+1}} H^{(i)} \tag{5}$$

$$X^{(i+1)} = X^{(i)} + h^{(i)}_{J_i} \cdot \frac{b^T_{I_{i+1}} - a^T_{I_{i+1}} X^{(i)}}{s^{(i)}_{J_i}} \tag{6}$$

$$\hat{H}^{(i)} = H^{(i)} - \frac{h^{(i)}_{J_i}}{s^{(i)}_{J_i}} s^{(i)T} \tag{7}$$

$$H^{(i+1)} = \begin{pmatrix} \hat{h}^{(i)}_0 & \cdots & \hat{h}^{(i)}_{J_i-1} & \hat{h}^{(i)}_{J_i+1} & \cdots & \hat{h}^{(i)}_{n-i} \end{pmatrix} \tag{8}$$

where $h^{(i)}_{J_i}$ is the J_i-th column of $H^{(i)}$, $s^{(i)}_{J_i}$ is the J_i-th element of $s^{(i)T}$ and J_i is, in principle, chosen such that $h^{(i)}_{J_i}$ has to lowest number of non-zero elements under the condition that $s^{(i)}_{J_i} \neq 0$. In practice, this condition has to be modified to exclude too small elements for reasons of numerical stability, leading to $|s^{(i)}_{J_i}| > \max(\epsilon, \tau \cdot \|s^{(i)T}\|_\infty)$. Here, ϵ is a small constant depending on the machine precision of the number type used and the problem size, $\|s^{(i)T}\|_\infty = \max_j |s^{(i)}_j|$, and $0 \leq \tau < 1$ allows balancing sparsity preservation and numerical stability.

After the last iteration, $X_\mathrm{p} = X^{(m)}$ and $H = H^{(m)}$ describe a solution in the form of eq. (2).

2.1. Proof of correctness

While a detailed analysis is left to [1], a proof of the algorithm's correctness is given here for completeness. Let

$$A^{(i)} = \begin{pmatrix} a^T_{I_1} \\ \vdots \\ a^T_{I_i} \end{pmatrix} \qquad B^{(i)} = \begin{pmatrix} b^T_{I_1} \\ \vdots \\ b^T_{I_i} \end{pmatrix}. \tag{9}$$

We will show by induction that, for every $i \in 0, \dots, m$, the conditions

$$A^{(i)} X^{(i)} = B^{(i)} \tag{10}$$

$$A^{(i)} H^{(i)} = \mathbf{0} \tag{11}$$

$$H^{(i)} \in \mathbb{R}^{n \times (n-i)} \text{ has linearly independent columns} \tag{12}$$

hold. As $A^{(m)}$ und $B^{(m)}$ become A and B by row permutation, eq. (10) implies that $X_\mathrm{p} = X^{(m)}$ is a specific solution of eq. (1) and eqs. (11) and (12) imply that the columns of $H = H^{(m)}$ span the nullspace of A.

For $i = 0$, the equation systems of eqs. (10) and (11) each comprise zero equations, whereby we consider them fulfilled. Equation (12) also holds trivially for $i = 0$ as $H^{(0)} = I$. It remains to show that if eqs. (10) to (12) hold for i, they also also for $i + 1$.

Substituting eq. (6) and $s_{J_i}^{(i)} = \boldsymbol{a}_{I_i}^T \boldsymbol{h}_{J_i}^{(i)}$, we obtain

$$\boldsymbol{A}^{(i+1)}\boldsymbol{X}^{(i+1)} = \begin{pmatrix} \boldsymbol{A}^{(i)} \\ \boldsymbol{a}_{I_{i+1}}^T \end{pmatrix} \cdot \left(\boldsymbol{X}^{(i)} + \boldsymbol{h}_{J_i}^{(i)} \cdot \frac{\boldsymbol{b}_{I_{i+1}}^T - \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{X}^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \right)$$

$$= \begin{pmatrix} \boldsymbol{B}^{(i)} \\ \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{X}^{(i)} + \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)} \cdot \frac{\boldsymbol{b}_{I_{i+1}}^T - \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{X}^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \end{pmatrix} =$$

$$\begin{pmatrix} \boldsymbol{B}^{(i)} \\ \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{X}^{(i)} + \boldsymbol{b}_{I_{i+1}}^T - \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{X}^{(i)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{(i)} \\ \boldsymbol{b}_{I_{i+1}}^T \end{pmatrix} = \boldsymbol{B}^{(i+1)} \quad (13)$$

by using eq. (10) and $\boldsymbol{A}^{(i)}\boldsymbol{h}_{J_i}^{(i)} = \boldsymbol{0}$, the J_i-th column von eq. (11). Similar substitution of eq. (7) leads to

$$\boldsymbol{A}^{(i+1)}\hat{\boldsymbol{H}}^{(i)} = \begin{pmatrix} \boldsymbol{A}^{(i)} \\ \boldsymbol{a}_{I_{i+1}}^T \end{pmatrix} \cdot \left(\boldsymbol{H}^{(i)} - \frac{\boldsymbol{h}_{J_i}^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \cdot \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{H}^{(i)} \right) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{H}^{(i)} - \frac{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \cdot \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{H}^{(i)} \end{pmatrix} = \boldsymbol{0} \quad (14)$$

using eq. (11) and its J_i-th column $\boldsymbol{A}^{(i)}\boldsymbol{h}_{J_i}^{(i)} = \boldsymbol{0}$. Then obviously also $\boldsymbol{A}^{(i+1)}\boldsymbol{H}^{(i+1)} = \boldsymbol{0}$.

For the linear independence of the columns of $\boldsymbol{H}^{(i+1)}$, we examine the $n - i$ columns of $\hat{\boldsymbol{H}}^{(i)}$ with respect to linear dependence, i.e. we try to find coefficients α_j such that

$$\sum_{j=1}^{n-i} \alpha_j \hat{\boldsymbol{h}}_j^{(i)} = \boldsymbol{0}. \quad (15)$$

Column-wise substitution of eq. (7) leads to

$$\sum_{j=1}^{n-i} \alpha_j \hat{\boldsymbol{h}}_j^{(i)} = \sum_{j=1}^{n-i} \alpha_j \left(\boldsymbol{h}_j^{(i)} - \frac{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_j^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \boldsymbol{h}_{J_i}^{(i)} \right) = \alpha_{J_i} \boldsymbol{h}_{J_i}^{(i)} + \sum_{\substack{j=1 \\ j \neq J_i}}^{n-i} \alpha_j \boldsymbol{h}_j^{(i)} - \alpha_{J_i} \boldsymbol{h}_{J_i}^{(i)} - \sum_{\substack{j=1 \\ j \neq J_i}}^{n-i} \alpha_j \frac{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_j^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \boldsymbol{h}_{J_i}^{(i)}$$

$$= \sum_{\substack{j=1 \\ j \neq J_i}}^{n-i} \alpha_j \boldsymbol{h}_j^{(i)} - \left(\sum_{\substack{j=1 \\ j \neq J_i}}^{n-i} \alpha_j \frac{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_j^{(i)}}{\boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)}} \right) \boldsymbol{h}_{J_i}^{(i)} = \boldsymbol{0}. \quad (16)$$

As all $\boldsymbol{h}_j^{(i)}$ are linearly independent, it follows that $\alpha_j = 0$ for $j \neq J_i$, but α_{J_i} ist arbitrary. Hence all columns of $\hat{\boldsymbol{H}}^{(i)}$ are linearly independent, exepect for the J_i-th one, which are exactly the columns of $\boldsymbol{H}^{(i+1)}$. (And $\hat{\boldsymbol{h}}_{J_i} = \boldsymbol{0}$.)

It should be noted that correctness is given irrespective of the permutation I_i and the only requirement concerning J_i is that $s_{J_i}^{(i)} = \boldsymbol{a}_{I_{i+1}}^T \boldsymbol{h}_{J_i}^{(i)} \neq 0$. At least one such column $\boldsymbol{h}_{J_i}^{(i)}$ exists, as otherwise $\boldsymbol{H}^{(i)}$ would span a $(n - i)$-dimensional nullspace of the $(i + 1) \times n$ matrix $\boldsymbol{A}^{(i+1)}$. But then $\boldsymbol{A}^{(i+1)}$ and hence $\boldsymbol{A}$ could not have independent rows, violating that assumption.

A. Example

The algorithm will now be applied to an example problem taken from [2]. There, a circuit is analyzed which leads to the difference equation

$$\begin{pmatrix} \boldsymbol{M}_\mathrm{v} & \boldsymbol{M}_\mathrm{i} & \bar{\boldsymbol{M}}_{\mathrm{x}'} & \boldsymbol{M}_\mathrm{q} \\ \boldsymbol{T}_\mathrm{v} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_\mathrm{i} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{M}}_\mathrm{x} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \bar{\boldsymbol{x}}(n-1) + \begin{pmatrix} \bar{\boldsymbol{u}}(n) \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \tag{17}$$

where

$$\boldsymbol{M}_\mathrm{v} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{M}_\mathrm{i} = \begin{pmatrix} R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

$$\bar{\boldsymbol{M}}_{\mathrm{x}'} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ -f_\mathrm{s} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\boldsymbol{M}}_\mathrm{x} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -f_\mathrm{s} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{M}_\mathrm{q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \bar{\boldsymbol{u}}(n) = \begin{pmatrix} 0 \\ \bar{u}_\mathrm{in}(n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

$$\boldsymbol{T}_\mathrm{v} = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \quad \boldsymbol{T}_\mathrm{i} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{20}$$

are derived (in a systematic fashion) from the schematics. Plugging these matrices into eq. (17) we get

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -f_\mathrm{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -f_\mathrm{s} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \bar{\boldsymbol{x}}(n-1) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \bar{u}_\mathrm{in}(n). \quad (21)$$

To ease notation in the following, we first re-order the rows to have an increasing number of non-zero elements in the rows of the LHS matrix. Then we do not need a permutation later, i.e. we may use $I_i = i$. To further ease manual computatation, of all the rows with two non-zero elements, we sort those with only ± 1 before the others. This then gives

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -f_{\mathrm{s}} & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -f_{\mathrm{s}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \bar{\boldsymbol{x}}(n-1) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \bar{u}_{\mathrm{in}}(n). \tag{22}$$

In the following, we let

$$\boldsymbol{A} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -f_{\mathrm{s}} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{23}$$

and

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \\ -f_\mathrm{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{24}$$

so that the system becomes

$$\boldsymbol{A} \cdot \begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix} = \boldsymbol{B} \cdot \begin{pmatrix} \bar{\boldsymbol{x}}(n-1) \\ \bar{u}_\mathrm{in}(n) \end{pmatrix}. \tag{25}$$

We now apply the algorithm to $\boldsymbol{A}$ and $\boldsymbol{B}$ to obtain $\boldsymbol{X}$ and $\boldsymbol{H}$. Due to the sizes of the involved matrices and the number of iterations ($m = 13$), this will take up a lot of space. But note that thanks to the sparsity, the number of actual computations to be performed is actually quite low.

For initialization, we let

$$\boldsymbol{H}^{(0)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad \boldsymbol{X}^{(0)} = \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{pmatrix}. \tag{26}$$

In the first iteration, we have

$$\boldsymbol{s}^{(0)T} = \boldsymbol{a}_1^T \boldsymbol{H}^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

so that $J_0 = 2$ is our only choice. Hence we use

$$\boldsymbol{h}_2^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{28}$$

and

$$s_2^{(0)} = \boldsymbol{a}_1^T \boldsymbol{h}_2^{(0)} = 1. \tag{29}$$

This gives

$$\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \boldsymbol{h}_2^{(0)} \cdot \frac{\boldsymbol{b}_1^T - \boldsymbol{a}_1^T \boldsymbol{X}^{(0)}}{s_2^{(0)}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix}}{1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{30}$$

and

$$\hat{\boldsymbol{H}}^{(0)} = \boldsymbol{H}^{(0)} - \frac{\boldsymbol{h}_2^{(0)}}{s_2^{(0)}} \boldsymbol{s}^{(0)T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

as $\boldsymbol{h}_2^{(0)} \boldsymbol{s}^{(0)T}$ has only a single non-zero entry, namely a 1 in row 2, column 2. Dropping the second column then gives

$$
\boldsymbol{H}^{(1)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{32}
$$

In the second iteration, we then have

$$
\boldsymbol{a}_2^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{33}
$$

and therefore

$$
\boldsymbol{s}^{(1)T} = \boldsymbol{a}_2^T \boldsymbol{H}^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{34}
$$

and pick $J_1 = 3$. Hence we use

$$
\boldsymbol{h}_3^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{35}
$$

and

$$
s_3^{(1)} = \boldsymbol{a}_2^T \boldsymbol{h}_3^{(1)} = 1. \tag{36}
$$

Observe that $\boldsymbol{b}_2^T = \boldsymbol{b}_3^T = \cdots = \boldsymbol{b}_8^T = \begin{pmatrix} 0 & 0 \end{pmatrix}$ so that

$$
\boldsymbol{X}^{(i+1)} = \boldsymbol{X}^{(i)} + \boldsymbol{h}_{J_i}^{(i)} \cdot \frac{\boldsymbol{b}_{i+1}^T - \boldsymbol{a}_{i+1}^T \boldsymbol{X}^{(i)}}{s_{J_i}^{(i)}} = \boldsymbol{X}^{(i)} - \boldsymbol{h}_{J_i}^{(i)} \cdot \frac{\boldsymbol{a}_{i+1}^T \boldsymbol{X}^{(i)}}{s_{J_i}^{(i)}} \tag{37}
$$

for $i = 1, \dots, 7$. Furthermore $\boldsymbol{a}_2^T \boldsymbol{X}^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$ so that $\boldsymbol{X}^{(2)} = \boldsymbol{X}^{(1)}$. But this likewise applies for $i = 2, \dots, 7$, so that $\boldsymbol{X}^{(8)} = \cdots = \boldsymbol{X}^{(2)} = \boldsymbol{X}^{(1)}$ and we can focus on $\boldsymbol{H}^{(i)}$ in the following. We now have

$$
\frac{\boldsymbol{h}_3^{(1)}}{s_3^{(1)}} \boldsymbol{s}^{(1)T} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{38}
$$

so that

$$\hat{\boldsymbol{H}}^{(1)} = \boldsymbol{H}^{(1)} - \frac{\boldsymbol{h}_3^{(1)}}{s_3^{(1)}}\boldsymbol{s}^{(1)T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{39}$$

and after dropping the third column

$$\boldsymbol{H}^{(2)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{40}$$

In the third iteration, with

$$\boldsymbol{a}_3^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{41}$$

we then have

$$\boldsymbol{s}^{(2)T} = \boldsymbol{a}_3^T \boldsymbol{H}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{42}$$

and pick $J_2 = 7$. Hence we use

$$\boldsymbol{h}_7^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{43}$$

and

$$s_7^{(2)} = \boldsymbol{a}_3^T \boldsymbol{h}_7^{(2)} = 1. \tag{44}$$

Now

$$\frac{\boldsymbol{h}_7^{(2)}}{s_7^{(2)}} \boldsymbol{s}^{(2)T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

and hence

$$\hat{\boldsymbol{H}}^{(2)} = \boldsymbol{H}^{(2)} - \frac{\boldsymbol{h}_7^{(2)}}{s_7^{(2)}} \boldsymbol{s}^{(2)T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{46}$$

Dropping the seventh column then gives

$$\boldsymbol{H}^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

In the fourth iteration, with

$$\boldsymbol{a}_4^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{48}$$

we find

$$\boldsymbol{s}^{(3)T} = \boldsymbol{a}_4^T \boldsymbol{H}^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{49}$$

and pick $J_3 = 3$. Hence we use

$$\boldsymbol{h}_3^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{50}$$

and

$$s_3^{(3)} = \boldsymbol{a}_4^T \boldsymbol{h}_3^{(3)} = 1. \tag{51}$$

Now

$$\frac{\boldsymbol{h}_3^{(3)}}{s_3^{(3)}} \boldsymbol{s}^{(3)T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

so that

$$\hat{\boldsymbol{H}}^{(3)} = \boldsymbol{H}^{(3)} - \frac{\boldsymbol{h}_3^{(3)}}{s_3^{(3)}}\boldsymbol{s}^{(3)T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{53}$$

and after dropping the third column

$$\boldsymbol{H}^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{54}$$

In the fifth iteration, given

$$\boldsymbol{a}_5^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{55}$$

we find

$$\boldsymbol{s}^{(4)T} = \boldsymbol{a}_5^T \boldsymbol{H}^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{56}$$

and pick $J_4 = 6$. Hence we use

$$\boldsymbol{h}_6^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{57}$$

and

$$s_6^{(4)} = \boldsymbol{a}_5^T \boldsymbol{h}_6^{(4)} = 1 \tag{58}$$

to obtain

$$\frac{\boldsymbol{h}_6^{(4)}}{s_6^{(4)}} \boldsymbol{s}^{(4)T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{59}$$

which leads to

$$\hat{\boldsymbol{H}}^{(4)} = \boldsymbol{H}^{(4)} - \frac{\boldsymbol{h}_6^{(4)}}{s_6^{(4)}} \boldsymbol{s}^{(4)T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{60}$$

and after dropping the sixth column

$$\boldsymbol{H}^{(5)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}. \tag{61}$$

In the sixth iteration, with

$$\boldsymbol{a}_6^T = \begin{pmatrix} 0&0&0&0&0&1&1&0&0&0&0&0&0&0&0 \end{pmatrix} \tag{62}$$

we find

$$\boldsymbol{s}^{(5)T} = \boldsymbol{a}_6^T \boldsymbol{H}^{(5)} = \begin{pmatrix} 0&0&1&1&0&0&0&0&0&0 \end{pmatrix} \tag{63}$$

and pick $J_5 = 3$. Hence we use

$$\boldsymbol{h}_3^{(5)} = \begin{pmatrix} 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \end{pmatrix}^T \tag{64}$$

and

$$s_3^{(5)} = \boldsymbol{a}_6^T \boldsymbol{h}_3^{(5)} = 1 \tag{65}$$

to obtain

$$
\begin{aligned}
\hat{\boldsymbol{H}}^{(5)} &= \boldsymbol{H}^{(5)} - \frac{\boldsymbol{h}_3^{(5)}}{s_3^{(5)}} \boldsymbol{s}^{(5)T} \\
&= \boldsymbol{H}^{(5)} - \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned} \tag{66}
$$

and by dropping the third column

$$
\boldsymbol{H}^{(6)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{67}
$$

In the seventh iteration, with

$$
\boldsymbol{a}_7^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{68}
$$

we find

$$
\boldsymbol{s}^{(6)T} = \boldsymbol{a}_7^T \boldsymbol{H}^{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{69}
$$

and pick $J_6 = 7$. Hence we use

$$\boldsymbol{h}_7^{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T \tag{70}$$

and

$$s_4^{(6)} = \boldsymbol{a}_7^T \boldsymbol{h}_7^{(6)} = 1. \tag{71}$$

This is the first time that $\boldsymbol{h}_{J_i}^{(i)}$ has more than one non-zero entry (namely two, the same as if we had choosen $J_6 = 9$) and thus additional non-zero entries are introduced in

$$\begin{aligned} \hat{\boldsymbol{H}}^{(6)} &= \boldsymbol{H}^{(6)} - \frac{\boldsymbol{h}_7^{(6)}}{s_4^{(6)}} \boldsymbol{s}^{(6)T} \\ &= \boldsymbol{H}^{(6)} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{72}$$

Dropping the seventh column we get

$$\boldsymbol{H}^{(7)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{73}$$

For the eighth iteration, with

$$\boldsymbol{a}_8^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{74}$$

we find

$$\boldsymbol{s}^{(7)T} = \boldsymbol{a}_8^T \boldsymbol{H}^{(7)} = \begin{pmatrix} -1 & 0 & -R & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{75}$$

and pick $J_7 = 1$. We thus get

$$\boldsymbol{h}_1^{(7)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{76}$$

(note that $J_7 = 3$ would have benn less favorable as $\boldsymbol{h}_3^{(7)}$ has two non-zero entries) and

$$s_1^{(7)} = \boldsymbol{a}_8^T \boldsymbol{h}_1^{(7)} = -1. \tag{77}$$

This leads to

$$\begin{aligned}
\hat{\boldsymbol{H}}^{(7)} &= \boldsymbol{H}^{(7)} - \frac{\boldsymbol{h}_1^{(7)}}{s_1^{(7)}} \boldsymbol{s}^{(7)T} \\
&= \boldsymbol{H}^{(7)} - \begin{pmatrix}
1 & 0 & R & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -R & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned} \tag{78}$$

and after dropping the first column

$$\boldsymbol{H}^{(8)} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{79}$$

In the ninth iteration we have

$$\boldsymbol{a}_9^T = \begin{pmatrix} 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{80}$$

and so

$$\boldsymbol{s}^{(8)T} = \boldsymbol{a}_9^T \boldsymbol{H}^{(8)} = \begin{pmatrix} C & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \tag{81}$$

and pick $J_8 = 4$. We thus get

$$\boldsymbol{h}_4^{(8)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{82}$$

and

$$s_4^{(8)} = \boldsymbol{a}_9^T \boldsymbol{h}_4^{(8)} = -\tfrac{1}{2}. \tag{83}$$

It is time to consider $\boldsymbol{X}^{(i)}$ again, for which we find

$$\boldsymbol{X}^{(9)} = \boldsymbol{X}^{(8)} + \boldsymbol{h}_4^{(8)} \cdot \frac{\boldsymbol{b}_9^T - \boldsymbol{a}_9^T \boldsymbol{X}^{(8)}}{s_4^{(8)}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix}}{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{84}$$

For the nullspace, we get

$$\hat{\boldsymbol{H}}^{(8)} = \boldsymbol{H}^{(8)} - \frac{\boldsymbol{h}_4^{(8)}}{s_4^{(8)}} \boldsymbol{s}^{(8)T} =$$

$$\boldsymbol{H}^{(8)} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2C & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{85}$$

and after dropping the fourth column

$$\boldsymbol{H}^{(9)} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{86}$$

In the tenth iteration we have

$$\boldsymbol{a}_{10}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -f_\mathrm{s} & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{87}$$

and so

$$\boldsymbol{s}^{(9)T} = \boldsymbol{a}_{10}^T \boldsymbol{H}^{(9)} = \begin{pmatrix} -2Cf_\mathrm{s} & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{88}$$

and pick $J_9 = 3$. We thus get

$$\boldsymbol{h}_3^{(9)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{89}$$

and

$$s_3^{(9)} = \boldsymbol{a}_{10}^T \boldsymbol{h}_3^{(9)} = 1. \tag{90}$$

It follows that

$$\boldsymbol{X}^{(10)} = \boldsymbol{X}^{(9)} + \boldsymbol{h}_3^{(9)} \cdot \frac{\boldsymbol{b}_{10}^T - \boldsymbol{a}_{10}^T \boldsymbol{X}^{(9)}}{s_3^{(9)}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} -f_\text{s} & 0 \end{pmatrix} - \begin{pmatrix} f_\text{s} & 0 \end{pmatrix}}{1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_\text{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{91}$$

and

$$\hat{\boldsymbol{H}}^{(9)} = \boldsymbol{H}^{(9)} - \frac{\boldsymbol{h}_3^{(9)}}{s_3^{(9)}} \boldsymbol{s}^{(9)T}$$

$$= \boldsymbol{H}^{(9)} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2Cf_\text{s} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2Cf_\text{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{92}$$

and after dropping the third column

$$
\boldsymbol{H}^{(10)} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2Cf_\mathrm{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{93}
$$

For the eleventh iteration we have

$$
\boldsymbol{a}_{11}^T = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{94}
$$

and so

$$
\boldsymbol{s}^{(10)T} = \boldsymbol{a}_{11}^T \boldsymbol{H}^{(10)} = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \end{pmatrix} \tag{95}
$$

and pick $J_{10} = 3$. We thus get

$$
\boldsymbol{h}_3^{(10)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T \tag{96}
$$

and

$$
s_3^{(10)} = \boldsymbol{a}_{11}^T \boldsymbol{h}_3^{(10)} = -1. \tag{97}
$$

It follows that

$$\boldsymbol{X}^{(11)} = \boldsymbol{X}^{(10)} + \boldsymbol{h}_3^{(10)} \cdot \frac{\boldsymbol{b}_{11}^T - \boldsymbol{a}_{11}^T \boldsymbol{X}^{(10)}}{s_3^{(10)}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_\mathrm{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix}}{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_\mathrm{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{98}$$

and

$$\hat{\boldsymbol{H}}^{(11)} = \boldsymbol{H}^{(10)} - \frac{\boldsymbol{h}_3^{(10)}}{s_3^{(10)}} \boldsymbol{s}^{(10)T} = \boldsymbol{H}^{(10)} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2Cf_\mathrm{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{99}$$

and after dropping the third column

$$\boldsymbol{H}^{(11)} = \begin{pmatrix} 0 & -R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2Cf_\mathrm{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{100}$$

For the twelvth iteration we have

$$\boldsymbol{a}_{12}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{101}$$

and so

$$\boldsymbol{s}^{(11)T} = \boldsymbol{a}_{12}^T \boldsymbol{H}^{(11)} = \begin{pmatrix} 2Cf_\mathrm{s} & 1 & 0 & 1 \end{pmatrix} \tag{102}$$

and pick $J_{11} = 2$. We thus get

$$\boldsymbol{h}_2^{(11)} = \begin{pmatrix} -R & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{103}$$

and

$$s_2^{(11)} = \boldsymbol{a}_{12}^T \boldsymbol{h}_2^{(11)} = 1. \tag{104}$$

It follows that

$$\boldsymbol{X}^{(12)} = \boldsymbol{X}^{(11)} + \boldsymbol{h}_2^{(11)} \cdot \frac{\boldsymbol{b}_{12}^T - \boldsymbol{a}_{12}^T \boldsymbol{X}^{(11)}}{s_2^{(11)}}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_\mathrm{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -R \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} -2f_\mathrm{s} & 0 \end{pmatrix}}{1} = \begin{pmatrix} -2Rf_\mathrm{s} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_\mathrm{s} & 0 \\ 2f_\mathrm{s} & 0 \\ -2f_\mathrm{s} & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (105)$$

and

$$\hat{\boldsymbol{H}}^{(11)} = \hat{\boldsymbol{H}}^{(11)} - \frac{\boldsymbol{h}_2^{(11)}}{s_2^{(11)}} \boldsymbol{s}^{(11)T} = \hat{\boldsymbol{H}}^{(11)} - \begin{pmatrix} -2RCf_\mathrm{s} & -R & 0 & -R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2Cf_\mathrm{s} & -1 & 0 & -1 \\ 2Cf_\mathrm{s} & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2RCf_\mathrm{s} & 0 & 0 & R \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 2Cf_\mathrm{s} & 0 & 0 & 1 \\ -2Cf_\mathrm{s} & 0 & 0 & -1 \\ 2Cf_\mathrm{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2C & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (106)$$

and after dropping the second column

$$\boldsymbol{H}^{(12)} = \begin{pmatrix} 2RCf_s & 0 & R \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2Cf_s & 0 & 1 \\ -2Cf_s & 0 & -1 \\ 2Cf_s & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 2C & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{107}$$

In the final iteration we have

$$\boldsymbol{a}_{13}^T = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{108}$$

and so

$$\boldsymbol{s}^{(11)T} = \boldsymbol{a}_{13}^T \boldsymbol{H}^{(12)} = \begin{pmatrix} 2RCf_s + 1 & 0 & R \end{pmatrix} \tag{109}$$

and pick $J_{12} = 3$. We thus get

$$\boldsymbol{h}_3^{(12)} = \begin{pmatrix} R & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}^T \tag{110}$$

and

$$s_3^{(12)} = \boldsymbol{a}_{13}^T \boldsymbol{h}_3^{(12)} = R. \tag{111}$$

It follows that

$$
\begin{aligned}
\boldsymbol{X}^{(13)} &= \boldsymbol{X}^{(12)} + \boldsymbol{h}_3^{(12)} \cdot \frac{\boldsymbol{b}_{13}^T - \boldsymbol{a}_{13}^T \boldsymbol{X}^{(12)}}{s_3^{(12)}} \\
&= \begin{pmatrix} -2Rf_s & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2f_s & 0 \\ 2f_s & 0 \\ -2f_s & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} R \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{\begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} -2Rf_s & -1 \end{pmatrix}}{R} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{R} \\ 0 & -\frac{1}{R} \\ -2f_s & 0 \\ 2f_s & \frac{1}{R} \\ 2f_s & \frac{1}{R} \\ -1 & 0 \\ 0 & 0 \\ 2f_s & \frac{1}{R} \\ 0 & 0 \\ 2f_s & \frac{1}{R} \end{pmatrix}
\end{aligned} \tag{112}
$$

and

$$
\hat{\boldsymbol{H}}^{(12)} = \hat{\boldsymbol{H}}^{(12)} - \frac{\boldsymbol{h}_3^{(12)}}{s_3^{(12)}} \boldsymbol{s}^{(11)T} = \hat{\boldsymbol{H}}^{(12)} - \begin{pmatrix} 2RCf_s + 1 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2RCf_s+1}{R} & 0 & 1 \\ -\frac{2RCf_s+1}{R} & 0 & -1 \\ 0 & 0 & 0 \\ \frac{2RCf_s+1}{R} & 0 & 1 \\ \frac{2RCf_s+1}{R} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2RCf_s+1}{R} & 0 & 1 \\ 0 & 0 & 0 \\ \frac{2RCf_s+1}{R} & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{R} & 0 & 0 \\ \frac{1}{R} & 0 & 0 \\ 2Cf_s & 0 & 0 \\ -\frac{2RCf_s+1}{R} & 0 & 0 \\ -\frac{2RCf_s+1}{R} & 0 & 0 \\ 2C & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{2RCf_s+1}{R} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2RCf_s+1}{R} & 0 & 0 \end{pmatrix} \tag{113}
$$

and after dropping the third column

$$\boldsymbol{H}^{(13)} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ -\frac{1}{R} & 0 \\ \frac{1}{R} & 0 \\ 2Cf_\mathrm{s} & 0 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ 2C & 0 \\ 1 & -1 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ 0 & 1 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \end{pmatrix}. \tag{114}$$

The solution

$$\begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix} = \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & \frac{1}{R} \\ 0 & -\frac{1}{R} \\ -2f_\mathrm{s} & 0 \\ 2f_\mathrm{s} & \frac{1}{R} \\ 2f_\mathrm{s} & \frac{1}{R} \\ \hline -1 & 0 \\ \hline 0 & 0 \\ 2f_\mathrm{s} & \frac{1}{R} \\ 0 & 0 \\ 2f_\mathrm{s} & \frac{1}{R} \end{array}\right) \begin{pmatrix} \bar{\boldsymbol{x}}(n-1) \\ \bar{u}_\mathrm{in}(n) \end{pmatrix} + \left(\begin{array}{cc} -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ \hline -\frac{1}{R} & 0 \\ \frac{1}{R} & 0 \\ 2Cf_\mathrm{s} & 0 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ \hline 2C & 0 \\ \hline 1 & -1 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \\ 0 & 1 \\ -\frac{2RCf_\mathrm{s}+1}{R} & 0 \end{array}\right) \boldsymbol{z}(n) \tag{115}$$

thus obtained is indeed equal to the one used in [2]. But note that neither the permutation I_i nor the choices for all J_i were uniquely determined, so other (but equally valid) solutions could have resulted.

References

[1] M. Khorramizadeh and N. Mahdavi-Amiri. “An efficient algorithm for sparse null space basis problem using ABS methods”. In: *Numerical Algorithms* 62.3 (June 2012), pp. 469–485.

[2] M. Holters and U. Zölzer. “A Generalized Method for the Derivation of Non-linear State-space Models from Circuit Schematics”. In: *Proceedings of the 23rd European Signal Processing Conference* (*EUSIPCO*). 2015, pp. 1078–1082.