SUPPLEMENTARY MATERIALS: PARALLEL-IN-TIME MULTIGRID WITH ADAPTIVE SPATIAL COARSENING FOR THE LINEAR ADVECTION AND INVISCID BURGERS EQUATIONS*

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SM1. Explicit Time-stepping for Case A4.

	$(\log_2(N_x), \log_2(N_t))$		
$\log_2(p_x)$	$(14, 15)$	$(15, 16)$	$(16, 17)$
2	831.35	—	—
3	434.53	1670.53	—
4	227.82	864.64	3353.94
5	125.45	448.79	1738.12
6	74.32	253.04	910.10
7	50.15	149.25	511.78
8	39.59	102.02	305.50
9	34.83	83.72	218.12
10	34.23	74.67	178.47
11	35.82	73.98	161.25
12	38.04	77.72	162.20

Table SM1

Strong scaling for serial explicit time-stepping with increasing amounts of spatial parallelism for fixed problem size.

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
3	iter	37	37	37	37	37	37	37
	time	2698.98	1878.07	1343.52	1050.28	847.13	745.30	685.69
4	iter	37	37	—	—	—	37	—
	time	1353.76	943.93	—	—	—	378.78	—
5	iter	37	—	37	—	37	—	—
	time	678.54	—	356.30	—	219.17	—	—
6	iter	37	—	—	37	—	—	—
	time	350.12	—	—	136.98	—	—	—
7	iter	37	—	37	—	37	—	—
	time	177.02	—	93.50	—	57.69	—	—
8	iter	37	37	—	—	—	37	—
	time	105.31	88.72	—	—	—	27.56	—
9	iter	37	—	—	—	—	—	37
	time	58.28	—	—	—	—	—	15.49

Table SM2

Original Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{14}, 2^{15})$.

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$\log_2$(pt)	$\log_2$(px)	3	4	5	6	7	8	9
4	iter	39	39	39	39	39	39	39
	time	3374.07	2271.85	1496.64	1151.49	932.99	822.80	758.01
5	iter	39	39	—	—	—	39	—
	time	1784.16	1135.81	—	—	—	417.14	—
6	iter	39	—	39	—	39	—	—
	time	909.66	—	395.43	—	240.30	—	—
7	iter	39	—	—	39	—	—	—
	time	458.25	—	—	150.03	—	—	—
8	iter	39	—	39	—	39	—	—
	time	237.21	—	104.18	—	63.86	—	—
9	iter	39	39	—	—	—	39	—
	time	123.08	81.64	—	—	—	31.99	—
10	iter	39	—	—	—	—	—	—
	time	66.88	—	—	—	—	—	—

Table SM3

Doubled Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{15}, 2^{16})$.

$\log_2$(pt)	$\log_2$(px)	4	5	6	7	8	9
5	iter	44	44	44	44	44	44
	time	4395.27	2702.93	1838.73	1362.48	1098.43	983.13
6	iter	44	44	—	—	44	—
	time	2243.41	1359.52	—	—	569.38	—
7	iter	44	—	44	44	—	—
	time	1148.06	—	464.86	349.17	—	—
8	iter	44	—	44	44	—	—
	time	578.08	—	237.94	177.53	—	—
9	iter	44	44	—	—	44	—
	time	295.32	184.29	—	—	78.33	—
10	iter	44	—	—	—	—	—
	time	156.64	—	—	—	—	—

Table SM4

Quadrupled Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{16}, 2^{17})$.

SM2. Implicit Time-stepping for Case A4.

	$(\log_2(N_x), \log_2(N_t))$
$\log_2(p_x)$	$(14, 14)$
2	782.02
3	518.83
4	394.58
5	357.26
6	336.55
7	348.35
8	359.36

TABLE SM5

Results for serial implicit time-stepping with increasing spatial parallelism, $(N_x, N_t) = (2^{14}, 2^{14})$.

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
2	iter	46	46	46	46	46	46	46
	time	8230.19	6686.69	5728.51	5202.09	4745.97	4498.12	4386.64
3	iter	46	46	—	—	—	46	—
	time	4106.38	3342.14	—	—	—	2260.45	—
4	iter	46	—	46	—	46	—	—
	time	2071.04	—	1496.78	—	1203.23	—	—
5	iter	46	—	—	46	—	—	—
	time	1075.52	—	—	669.24	—	—	—
6	iter	46	—	46	—	46	—	—
	time	546.51	—	398.02	—	313.18	—	—
7	iter	46	46	—	—	—	46	—
	time	312.58	277.75	—	—	—	157.05	—
8	iter	46	—	—	—	—	—	46
	time	180.42	—	—	—	—	—	86.90

TABLE SM6

Spatial Coarsening: *Strong scaling for implicit MGRIT F-cycles, $(N_x, N_t) = (2^{14}, 2^{14})$.*

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
2	iter	30	30	30	230	30	29	30
	time	6531.90	4547.23	3524.90	3104.28	2833.67	2599.01	2596.10
3	iter	30	30	—	—	—	30	—
	time	3261.04	2274.50	—	—	—	1348.17	—
4	iter	30	—	29	—	30	—	—
	time	1638.08	—	888.57	—	716.87	—	—
5	iter	30	—	—	30	—	—	—
	time	844.48	—	—	400.95	—	—	—
6	iter	30	—	30	—	30	—	—
	time	436.60	—	241.86	—	186.06	—	—
7	iter	30	30	—	—	—	30	—
	time	244.24	176.49	—	—	—	93.29	—
8	iter	30	—	—	—	—	—	29
	time	145.93	—	—	—	—	—	49.69

TABLE SM7

No Spatial Coarsening: *Strong scaling for implicit MGRIT F-cycles, $(N_x, N_t) = (2^{14}, 2^{14})$.*

SM3. Explicit Time-stepping for Case A5.

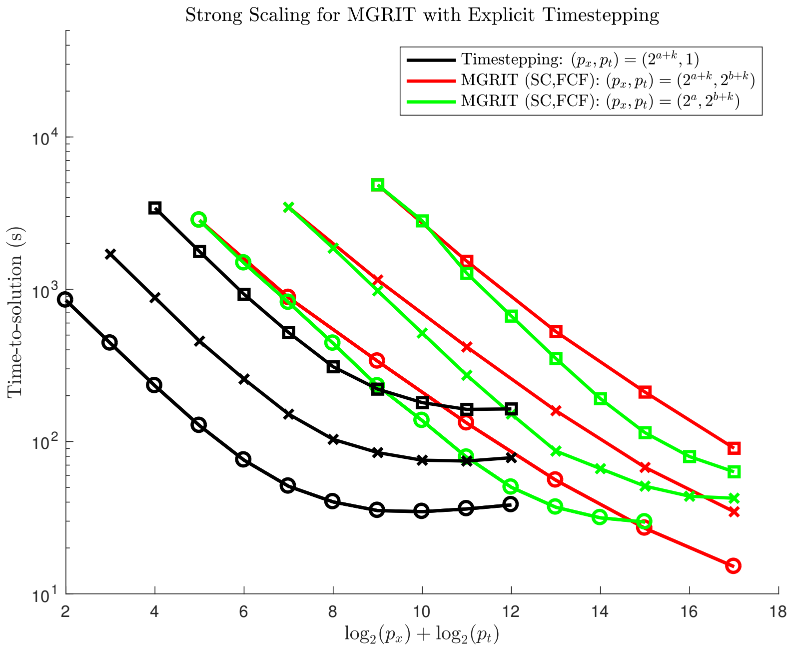

FIG. SM1. *Comparison of serial time-stepping with spatial parallelism to MGRIT with FCF relaxation and different combinations of space-time parallelism for three different problem sizes on up to 131072 cores, $(N_x, N_t) = (2^n, 2^{n+1})$. These results correspond to Tables SM9–SM12.*
○: $n = 14$, $a = 2$, $b = 3$. ×: $n = 15$, $a = 3$, $b = 4$. □: $n = 16$, $a = 4$, $b = 5$.

		(a,b,n)		
		$(2,3,14)$	$(3,4,15)$	$(4,5,16)$
(p_x, p_t)	$(2^{a+k}, 2^{b+k})$	2.30	2.15	1.78
	$(2^a, 2^{b+k})$	1.17	1.75	2.55

TABLE SM8
Best speedup achieved for explicit time-stepping strong scaling tests, $(N_x, N_t) = (2^n, 2^{n+1})$.

	$(\log_2(N_x), \log_2(N_t))$		
$\log_2(p_x)$	$(14,15)$	$(15,16)$	$(16,17)$
2	846.33	—	—
3	441.81	1699.65	—
4	232.14	880.60	3419.02
5	127.19	456.45	1769.27
6	75.44	257.11	926.55
7	50.81	151.31	519.89
8	40.13	103.31	309.69
9	35.19	84.62	220.80
10	34.63	75.39	180.07
11	36.10	74.48	162.61
12	38.36	78.33	163.74

TABLE SM9
Strong scaling for explicit time-stepping with increasing amounts of spatial parallelism and no temporal parallelism for $a(x,t) = -\sin(2.5\pi t)\sin(\pi x)$ and fixed problem size.

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
3	iter	36	36	36	36	36	36	36
	time	2836.37	1691.60	1207.62	986.34	809.76	714.02	647.39
4	iter	36	36	—	—	—	36	—
	time	1484.32	880.73	—	—	—	365.42	—
5	iter	36	—	36	—	36	—	—
	time	818.17	—	336.45	—	210.60	—	—
6	iter	36	—	—	36	—	—	—
	time	440.94	—	—	132.10	—	—	—
7	iter	36	—	36	—	36	—	—
	time	231.56	—	91.40	—	55.70	—	—
8	iter	36	36	—	—	—	36	—
	time	137.10	90.40	—	—	—	26.92	—
9	iter	36	—	—	—	—	—	36
	time	78.75	—	—	—	—	—	15.08

TABLE SM10

Original Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{14}, 2^{15})$.

$\log_2$(pt)	$\log_2$(px)	3	4	5	6	7	8	9
4	iter	42	41	41	42	41	42	41
	time	3455.59	2216.87	1493.09	1209.49	973.17	875.18	784.65
5	iter	41	41	—	—	—	41	—
	time	1860.44	1150.40	—	—	—	438.65	—
6	iter	41	—	42	—	42	—	—
	time	976.22	—	417.92	—	258.22	—	—
7	iter	42	—	—	41	—	—	—
	time	515.44	—	—	159.37	—	—	—
8	iter	41	—	41	—	41	—	—
	time	272.32	—	110.88	—	67.64	—	—
9	iter	42	42	—	—	—	42	—
	time	151.43	91.31	—	—	—	34.59	—
10	iter	41	—	—	—	—	—	—
	time	86.61	—	—	—	—	—	—

TABLE SM11

Doubled Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{15}, 2^{16})$.

$\log_2$(pt)	$\log_2$(px)	4	5	6	7	8	9
5	iter	51	52	50	51	50	50
	time	4846.38	3999.70	3592.11	3410.69	3301.74	3500.35
6	iter	50	51	—	—	51	—
	time	2811.06	1526.86	—	—	1735.72	—
7	iter	50	—	50	51	—	—
	time	1271.31	—	526.61	409.13	—	—
8	iter	51	—	51	51	—	—
	time	666.09	—	278.37	211.18	—	—
9	iter	51	51	—	—	50	—
	time	350.51	215.74	—	—	90.34	—
10	iter	50	—	—	—	—	—
	time	191.07	—	—	—	—	—

TABLE SM12

Quadrupled Size: *Strong scaling for explicit MGRIT F-cycles,* $(N_x, N_t) = (2^{16}, 2^{17})$.

					Original			Oscillatory		
Trial	$\log_2(N_x)$	$\log_2(N_t)$	$\log_2(p_x)$	$\log_2(p_t)$	ξ	Iter	Time	ξ	Iter	Time
1	10	11	0	1	1/4	30	184.86	16	30	184.98
2	11	12	1	2	1/4	31	223.12	32	31	222.95
3	12	13	2	3	1/4	33	245.39	64	33	245.68
4	13	14	3	4	1/4	35	291.70	128	35	291.61
5	14	15	4	5	1/4	36	339.98	256	36	339.53

Table SM13

Weak scaling for explicit MGRIT F-cycles with $u_0(x) = \sin(2\pi\xi x)$ and increasing N_x and N_t.

					Original			Oscillatory		
Trial	$\log_2(N_x)$	$\log_2(N_t)$	$\log_2(p_x)$	$\log_2(p_t)$	ξ	Iter	Time	ξ	Iter	Time
1	10	11	0	1	1/4	30	184.63	16	30	185.09
2	10	12	0	2	1/4	14	111.16	16	15	117.20
3	10	13	0	3	1/4	13	110.60	16	14	116.89
4	10	14	0	4	1/4	10	94.61	16	11	101.02
5	10	15	0	5	1/4	8	83.49	16	9	90.12
6	10	16	0	6	1/4	6	72.14	16	6	72.20
7	10	17	0	7	1/4	7	82.17	16	7	82.09

Table SM14

Weak scaling for explicit MGRIT F-cycles with $u_0(x) = \sin(2\pi\xi x)$ and fixed N_x.

SM4. Implicit Time-stepping for Case A5.

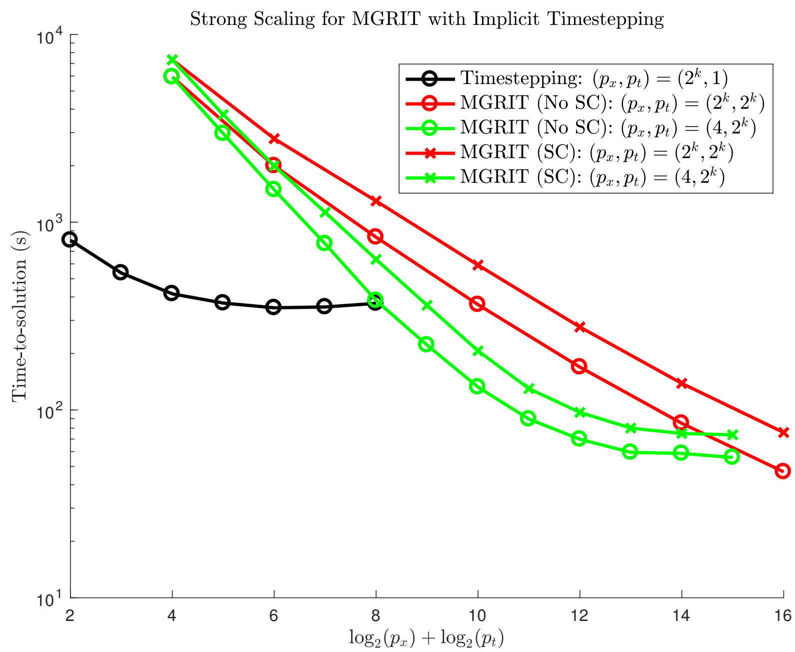

FIG. SM2. *Comparison of serial time-stepping with spatial parallelism to MGRIT using FCF relaxation with or without spatial coarsening for different combinations of space-time parallelism on up to 65536 cores. These results correspond to Tables SM16–SM18.*

		No SC	SC
(p_x, p_t)	$(2^k, 2^k)$	7.48	4.62
	$(2^4, 2^k)$	6.27	4.75

TABLE SM15
Best speedup achieved for implicit time-stepping strong scaling tests, $(N_x, N_t) = (2^{14}, 2^{14})$.

	$(\log_2(N_x), \log_2(N_t))$
$\log_2(p_x)$	(14, 14)
2	801.46
3	536.29
4	415.51
5	371.04
6	350.20
7	353.65
8	369.76

TABLE SM16
Results for serial implicit time-stepping with increasing spatial parallelism, $(N_x, N_t) = (2^{14}, 2^{14})$.

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
2	iter	27	26	27	27	27	26	27
	time	5937.06	3991.70	3210.82	2825.08	2578.00	2357.61	2364.47
3	iter	27	26	—	—	—	26	—
	time	2964.59	1997.37	—	—	—	1184.73	—
4	iter	27	—	27	—	26	—	—
	time	1488.67	—	832.33	—	629.40	—	—
5	iter	27	—	—	27	—	—	—
	time	767.72	—	—	364.10	—	—	—
6	iter	26	—	26	—	27	—	—
	time	382.61	—	211.93	—	169.23	—	—
7	iter	27	27	—	—	—	27	—
	time	221.78	160.31	—	—	—	84.93	—
8	iter	27	—	—	—	—	—	27
	time	132.43	—	—	—	—	—	46.83

TABLE SM17

No Spatial Coarsening: *Strong scaling for implicit MGRIT F-cycles,* $(N_x, N_t) = (2^{14}, 2^{14})$.

$\log_2$(pt)	$\log_2$(px)	2	3	4	5	6	7	8
2	iter	40	40	41	40	40	40	40
	time	7329.11	5463.71	4666.86	4374.57	4068.72	3883.04	3824.27
3	iter	40	40	—	—	—	40	—
	time	3740.71	2786.03	—	—	—	1964.83	—
4	iter	41	—	40	—	40	—	—
	time	1996.01	—	1297.06	—	1054.42	—	—
5	iter	40	—	—	40	—	—	—
	time	1129.15	—	—	591.32	—	—	—
6	iter	40	—	40	—	40	—	—
	time	634.29	—	354.25	—	276.62	—	—
7	iter	40	40	—	—	—	40	—
	time	361.17	264.02	—	—	—	138.59	—
8	iter	40	—	—	—	—	—	40
	time	206.53	—	—	—	—	—	75.87

TABLE SM18

Spatial Coarsening: *Strong scaling for implicit MGRIT F-cycles,* $(N_x, N_t) = (2^{14}, 2^{14})$.

					Original			Oscillatory		
Trial	$\log_2(N_x)$	$\log_2(N_t)$	$\log_2(p_x)$	$\log_2(p_t)$	ξ	Iter	Time	ξ	Iter	Time
1	10	10	0	0	1/4	24	295.32	16	24	296.61
2	11	11	1	1	1/4	26	626.30	32	26	624.78
3	12	12	2	2	1/4	29	871.47	64	29	869.50
4	13	13	3	3	1/4	34	1124.93	128	33	1091.52
5	14	14	4	4	1/4	41	1309.98	256	41	1295.42

TABLE SM19

Weak scaling for implicit MGRIT F-cycles with $u_0(x) = \sin(2\pi\xi x)$ *and increasing* N_x *and* N_t.

					Original			Oscillatory		
Trial	$\log_2(N_x)$	$\log_2(N_t)$	$\log_2(p_x)$	$\log_2(p_t)$	ξ	Iter	Time	ξ	Iter	Time
1	10	10	0	0	1/4	24	297.03	16	24	296.59
2	10	11	0	1	1/4	27	367.88	16	27	367.27
3	10	12	0	2	1/4	13	224.39	16	14	238.11
4	10	13	0	3	1/4	11	203.74	16	11	203.43
5	10	14	0	4	1/4	9	176.65	16	9	176.33
6	10	15	0	5	1/4	8	162.44	16	8	162.16
7	10	16	0	6	1/4	6	133.52	16	7	148.48
8	10	17	0	7	1/4	5	120.45	16	5	120.26
9	10	18	0	8	1/4	4	107.93	16	4	107.84

Table SM20

Weak scaling for implicit MGRIT F-cycles with $u_0(x) = \sin(2\pi\xi x)$ *and fixed* N_x.

					Original			Oscillatory		
Trial	$\log_2(N_x)$	$\log_2(N_t)$	$\log_2(p_x)$	$\log_2(p_t)$	ξ	Iter	Time	ξ	Iter	Time
1	10	10	0	0	1/4	24	297.05	16	24	294.81
2	11	10	1	0	1/4	23	465.06	32	23	465.20
3	12	10	2	0	1/4	23	559.13	64	23	558.23
4	13	10	3	0	1/4	23	594.61	128	23	593.32
5	14	10	4	0	1/4	23	642.39	256	23	641.37
6	15	10	5	0	1/4	23	682.46	512	23	679.69
7	16	10	6	0	1/4	22	696.81	1024	23	721.51
8	17	10	7	0	1/4	22	750.16	2048	23	775.58
9	18	10	8	0	1/4	22	816.76	4096	22	812.83

Table SM21

Weak scaling for implicit MGRIT F-cycles with $u_0(x) = \sin(2\pi\xi x)$ *and fixed* N_t.