Semi Implicit Solver Stability in Gusto

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1 Introduction

We are trying to solve

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{f} \times \boldsymbol{u} + c_p \theta \boldsymbol{\nabla} \Pi + \boldsymbol{\nabla} \Phi = \boldsymbol{0}, \tag{1a}$$

$$\frac{\partial \rho_d}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_d \boldsymbol{u}) = 0, \tag{1b}$$

$$\frac{D\theta}{Dt} = 0. \tag{1c}$$

To do so, we discretise terms use a semi-implicit time discretisation, which results in the following linear problem in a quasi-Newton nonlinear solve

$$\mathcal{L}[\boldsymbol{x}^*]\boldsymbol{x'} = -\mathcal{R}(\boldsymbol{x^k}) \tag{2}$$

where $\boldsymbol{x} = [\boldsymbol{u}, \rho_d, \theta]$ and $\boldsymbol{x'} = \boldsymbol{x^{k+1}} - \boldsymbol{x^k}$. The residual term $\mathcal{R}\boldsymbol{x^k}$ contains explicit advection terms, α-weighted implicit-forcing terms and $(1-\alpha)$-weighted explicit forcing terms. The linear term $\mathcal{L}[\boldsymbol{x^*}]$ is a discretisation of a linearisation about a reference state $\boldsymbol{x^*}$ of the nonlinear equations. Here θ is eliminated and the equation for $\boldsymbol{u}$ and ρ_d is then solved using a hybridised method. $\mathcal{L}[\boldsymbol{x}^*]$ is then

$$\mathcal{L}[\boldsymbol{x}^*](\boldsymbol{x}) = \begin{pmatrix} \int \boldsymbol{w} \cdot \boldsymbol{u} d\boldsymbol{V} - c_p\tau_u\Delta t \int \boldsymbol{\nabla \cdot \theta_a V(w)}\Pi^* d\boldsymbol{V} + c_p\tau_u\Delta t \int [[\theta_a \boldsymbol{V}(\boldsymbol{w})]]_{\boldsymbol{n}} \langle \Pi^* \rangle dS \\ +c_p\tau_u\Delta t \int (\theta_a \boldsymbol{V}(\boldsymbol{w})) \cdot \boldsymbol{n} \langle \Pi^* \rangle ds - c_p\tau_u\Delta t \int \boldsymbol{\nabla \cdot \theta_a^* w}\Pi d\boldsymbol{V} + \boldsymbol{T_u}, \\ \\ \int \phi \rho_d d\boldsymbol{V} - \tau_\rho \Delta t \int (\boldsymbol{\nabla}\phi \cdot \boldsymbol{u})\rho_d^* d\boldsymbol{V} - \tau_\rho \Delta t \int [[\phi \boldsymbol{u}]]_{\boldsymbol{n}} \langle \rho_d^* \rangle dS + \int \phi \boldsymbol{u} \cdot \boldsymbol{n} \langle \rho_d^* \rangle ds + \boldsymbol{T_{\rho_d}} \end{pmatrix} \tag{3}$$

Where $\boldsymbol{T_u}$ and $\boldsymbol{T_{\rho_d}}$ are the trace terms:

$$\boldsymbol{T_u} = c_p\tau_u\Delta t \int [[\theta_a^* \boldsymbol{w}]]_{\boldsymbol{n}} \boldsymbol{l'} dS + c_p\tau_u\Delta t \int (\theta_a^* \boldsymbol{w}) \cdot \boldsymbol{n}\boldsymbol{l'} ds \tag{4}$$

$$\boldsymbol{T_{\rho_d}} = \int \lambda [[\boldsymbol{u}]]_n dS + \int \lambda (\boldsymbol{u} \cdot \boldsymbol{n}) ds \tag{5}$$

Here $\boldsymbol{w} \in \mathbb{W}_2$, $\lambda \in \mathbb{W}_{trace}$ and $\phi \in \mathbb{W}_3$ are the test functions, $d\boldsymbol{V}$ corresponds to integrating over a cell volume, dS to cell boundaries and ds external domain boundaries. x^* corresponds to a background/reference state. $[[\cdot]]_n$ is a jump and $\langle \cdot \rangle$ is an average. $\boldsymbol{V}(\boldsymbol{w})$ is the vertical component of the $\boldsymbol{w}$ test function. θ_a is the analytically eliminated θ variable calculated by

$$\theta_a = -\tau_\theta \Delta t \int (\boldsymbol{k} \cdot \boldsymbol{u})(\boldsymbol{k} \cdot \boldsymbol{\nabla}\theta^*) d\boldsymbol{V} + \mathcal{R}(\theta) \tag{6}$$

Finally, to recompute θ we have the linearisation

$$\mathcal{L}[\theta^*](\theta) = \int \gamma\theta d\boldsymbol{V} + \tau_\theta \Delta t \int \gamma(\boldsymbol{k}\cdot\boldsymbol{u})(\boldsymbol{k}\cdot\boldsymbol{\nabla}\theta^*)d\boldsymbol{V} \tag{7}$$

for the test function $\gamma \in \mathbb{W}_\theta$. Here τ_u, τ_ρ and τ_θ are the relaxation parameters.

The relaxation parameters can be selected to be the off centering α value, however setting $\tau_\rho = 1$ was found to improve convergence in [2]. Further, $\tau_\theta = 1$ was selected in [1] for similar reasons. At convergence, altering τ values does not change the discretisation. For a given solution, if we are not converging within the given 2 by 2 quasi-Newton solver loop, this may introduce instability or spurious oscillations. Therefore increasing nonlinear solver iterations or increasing the relaxation parameters may improve stability.

In addition, the implicit part of the residuals $\mathcal{R}(\theta)$ and $\mathcal{R}(\rho)$ can be set to zero in the inner loop, when the inner loop iteration count is greater than 1. This improves inner loop convergence as discussed in [1], as it removes the inconsistency that the implicit correction to the transport terms introduces.

References

[1] Thomas Melvin et al. “A mixed finite-element, finite-volume, semi-implicit discretisation for atmospheric dynamics: Spherical geometry”. In: *Quarterly Journal of the Royal Meteorological Society* (2024).

[2] Nigel Wood et al. “An inherently mass-conserving semi-implicit semi-Lagrangian discretization of the deep-atmosphere global non-hydrostatic equations”. In: *Quarterly Journal of the Royal Meteorological Society* 140.682 (2014), pp. 1505–1520.